Exercise 36

Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

Solution

Start by taking the derivative of the function with respect to x.

$$y' = \frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

$$= \frac{\left[\frac{d}{dx}(\ln x)\right]x - (\ln x)\left[\frac{d}{dx}(x)\right]}{x^2}$$

$$= \frac{\left(\frac{1}{x}\right)x - (\ln x)(1)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

The slopes of the tangent lines at $x = 1$ and $x = e$ are, respectively,

$$y'(1) = \frac{1 - \ln 1}{1^2} = \frac{1 - 0}{1^2} = 1$$

$$y'(e) = \frac{1 - \ln e}{e^2} = \frac{1 - 1}{e^2} = 0.$$

Therefore, the equations of the tangent lines at the points, $(1, 0)$ and $(e, 1/e)$, are as follows.

$$y - 0 = y'(1)(x - 1) \qquad\qquad y - \frac{1}{e} = y'(e)(x - e)$$

$$y - 0 = 1(x - 1) \qquad\qquad y - \frac{1}{e} = 0(x - e)$$

$$y = x - 1 \qquad\qquad y - \frac{1}{e} = 0$$

$$y = \frac{1}{e}$$

Below is a graph of the curve and its tangent lines at $(1, 0)$ and $(e, 1/e)$.